Noting that $n > 1$, the absolute value can be removed. The condition on n becomes $n - 1 > 1/0.01 = 100$, or $n > 101$. Thus, we take $N = 101$ or any larger number. This means that $|a_n - 1| < 0.01$ whenever $n > 101$.

b. Given *any* $\varepsilon > 0$, we must find a value of N (depending on ε) that guarantees that $|a_n - 1| = \left|\frac{n}{n-1} - 1\right| < \varepsilon$ whenever $n > N$. For $n > 1$ the inequality $\left|\frac{n}{n-1} - 1\right| < \varepsilon$ implies that

$$\left|\frac{n}{n-1} - 1\right| = \frac{1}{n-1} < \varepsilon.$$

Solving for n, we find that $\frac{1}{n-1} < \varepsilon$ or $n - 1 > \frac{1}{\varepsilon}$ or $n > \frac{1}{\varepsilon} + 1$. Therefore, given a tolerance $\varepsilon > 0$, we must look beyond a_N in the sequence, where $N \ge \frac{1}{\varepsilon} + 1$, to be sure that the terms of the sequence are within ε of the limit 1. Because we can provide a value of N for *any* $\varepsilon > 0$, the limit exists and equals 1. *Related Exercises 57–62* ◄

➤ In general, $1/\varepsilon + 1$ is not an integer, so N should be the least integer greater than $1/\varepsilon + 1$ or any larger integer.

SECTION 8.2 EXERCISES

Review Questions

1. Give an example of a nonincreasing sequence with a limit.
2. Give an example of a nondecreasing sequence without a limit.
3. Give an example of a bounded sequence that has a limit.
4. Give an example of a bounded sequence without a limit.
5. For what values of r does the sequence $\{r^n\}$ converge? Diverge?
6. Explain how the methods used to find the limit of a function as $x \to \infty$ are used to find the limit of a sequence.
7. Explain with a picture the formal definition of the limit of a sequence.
8. Explain how two sequences that differ only in their first ten terms can have the same limit.

Basic Skills

9–26. Limits of sequences *Find the limit of the following sequences or determine that the limit does not exist.*

9. $\left\{\frac{n^3}{n^4 + 1}\right\}$
10. $\left\{\frac{n^{12}}{3n^{12} + 4}\right\}$
11. $\left\{\frac{3n^3 - 1}{2n^3 + 1}\right\}$
12. $\left\{\frac{2e^{n+1}}{e^n}\right\}$
13. $\left\{\frac{\tan^{-1} n}{n}\right\}$
14. $\{n^{1/n}\}$
15. $\left\{\left(1 + \frac{2}{n}\right)^n\right\}$
16. $\left\{\left(\frac{n}{n+5}\right)^n\right\}$
17. $\left\{\sqrt{\left(1 + \frac{1}{2n}\right)^n}\right\}$
18. $\left\{\frac{\ln(1/n)}{n}\right\}$
19. $\left\{\left(\frac{1}{n}\right)^{1/n}\right\}$
20. $\left\{\left(1 - \frac{4}{n}\right)^n\right\}$
21. $\{b_n\}$ if $b_n = \begin{cases} n/(n+1) & \text{if } n \le 5000 \\ ne^{-n} & \text{if } n > 5000 \end{cases}$
22. $\{\ln(n^3 + 1) - \ln(3n^3 + 10n)\}$
23. $\{\ln \sin(1/n) + \ln n\}$
24. $\left\{\frac{\sin 6n}{5n}\right\}$
25. $\{n \sin(6/n)\}$
26. $\left\{\frac{n!}{n^n}\right\}$

27–34. Limits of sequences and graphing *Find the limit of the following sequences or determine that the limit does not exist. Verify your result with a graphing utility.*

27. $a_n = \sin\left(\frac{n\pi}{2}\right)$
28. $a_n = \frac{(-1)^n n}{n + 1}$
29. $a_n = \frac{\sin(n\pi/3)}{\sqrt{n}}$
30. $a_n = \frac{3^n}{3^n + 4^n}$
31. $a_n = e^{-n} \cos n$
32. $a_n = \frac{\ln n}{n^{1.1}}$
33. $a_n = (-1)^n \sqrt[n]{n}$
34. $a_n = \cot\left(\frac{n\pi}{2n + 2}\right)$

35–42. Geometric sequences *Determine whether the following sequences converge or diverge and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges.*

35. $\{0.2^n\}$
36. $\{1.2^n\}$
37. $\{(-0.7)^n\}$
38. $\{(-1.01)^n\}$
39. $\{1.00001^n\}$
40. $\{2^n 3^{-n}\}$
41. $\{(-2.5)^n\}$
42. $\{(-0.003)^n\}$

43–46. Squeeze Theorem *Find the limit of the following sequences or state that they diverge.*

43. $\left\{\frac{\sin n}{2^n}\right\}$
44. $\left\{\frac{\cos(n\pi/2)}{\sqrt{n}}\right\}$
45. $\left\{\frac{2\tan^{-1} n}{n^3 + 4}\right\}$
46. $\left\{\frac{n \sin^3 n}{n + 1}\right\}$

47. Periodic dosing Many people take aspirin on a regular basis as a preventive measure for heart disease. Suppose a person takes 80 mg of aspirin every 24 hr. Assume also that aspirin has a half-life of 24 hr; that is, every 24 hr half of the drug in the blood is eliminated.

a. Find a recurrence relation for the sequence $\{d_n\}$ that gives the amount of drug in the blood after the nth dose, where $d_1 = 80$.
b. Using a calculator, determine the limit of the sequence. In the long run, how much drug is in the person's blood?
c. Confirm the result of part (b) by finding the limit of $\{d_n\}$ directly.

48. A car loan Marie takes out a \$20,000 loan for a new car. The loan has an annual interest rate of 6% or, equivalently, a monthly interest rate of 0.5%. Each month, the bank adds interest to the loan balance (the interest is always 0.5% of the current balance), and then Marie makes a \$200 payment to reduce the loan balance. Let B_n be the loan balance immediately after the nth payment, where $B_0 = \$20{,}000$.

a. Write the first five terms of the sequence $\{B_n\}$.
b. Find a recurrence relation that generates the sequence $\{B_n\}$.
c. Determine how many months are needed to reduce the loan balance to zero.

49. A savings plan James begins a savings plan in which he deposits \$100 at the beginning of each month into an account that earns 9% interest annually or, equivalently, 0.75% per month. To be clear, on the first day of each month, the bank adds 0.75% of the current balance as interest, and then James deposits \$100. Let B_n be the balance in the account after the nth payment, where $B_0 = \$0$.

a. Write the first five terms of the sequence $\{B_n\}$.
b. Find a recurrence relation that generates the sequence $\{B_n\}$.
c. Determine how many months are needed to reach a balance of \$5000.

50. Diluting a solution Suppose a tank is filled with 100 L of a 40% alcohol solution (by volume). You repeatedly perform the following operation: Remove 2 L of the solution from the tank and replace them with 2 L of 10% alcohol solution.

a. Let C_n be the concentration of the solution in the tank after the nth replacement, where $C_0 = 40\%$. Write the first five terms of the sequence $\{C_n\}$.
b. After how many replacements does the alcohol concentration reach 15%?
c. Determine the limiting (steady-state) concentration of the solution that is approached after many replacements.

51–56. Comparing growth rates of sequences *Determine which sequence has the greater growth rate as $n \to \infty$. Be sure to justify and explain your work.*

51. $a_n = n^2$; $b_n = n^2 \ln n$
52. $a_n = 3^n$; $b_n = n!$
53. $a_n = 3n^n$; $b_n = 100n!$
54. $a_n = \ln(n^{12})$; $b_n = n^{1/2}$
55. $a_n = n^{1/10}$; $b_n = n^{1/2}$
56. $a_n = e^{n/10}$; $b_n = 2^n$

57–62. Formal proofs of limits *Use the formal definition of the limit of a sequence to prove the following limits.*

57. $\lim_{n\to\infty} \frac{1}{n} = 0$
58. $\lim_{n\to\infty} \frac{1}{n^2} = 0$
59. $\lim_{n\to\infty} \frac{3n^2}{4n^2 + 1} = \frac{3}{4}$
60. $\lim_{n\to\infty} b^{-n} = 0$, for $b > 1$
61. $\lim_{n\to\infty} \frac{cn}{bn + 1} = \frac{c}{b}$, for real numbers $c > 0$ and $b > 0$
62. $\lim_{n\to\infty} \frac{n}{n^2 + 1} = 0$

Further Explorations

63. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} b_n = 3$; then $\lim_{n\to\infty} \frac{b_n}{a_n} = 3$.
b. If $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \infty$; then $\lim_{n\to\infty} a_n b_n = 0$.
c. The convergent sequences $\{a_n\}$ and $\{b_n\}$ differ in their first 100 terms, but $a_n = b_n$ for $n > 100$. It follows that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.
d. If $\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$ and $\{b_n\} = \left\{1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \dots\right\}$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.
e. If the sequence $\{a_n\}$ converges, then the sequence $\{(-1)^n a_n\}$ converges.
f. If the sequence $\{a_n\}$ diverges, then the sequence $\{0.000001\, a_n\}$ diverges.

64–65. Reindexing *Express each sequence $\{a_n\}_{n=1}^{\infty}$ as an equivalent sequence of the form $\{b_n\}_{n=3}^{\infty}$.*

64. $\{2n + 1\}_{n=1}^{\infty}$
65. $\{n^2 + 6n - 9\}_{n=1}^{\infty}$

66–69. More sequences *Evaluate the limit of the following sequences.*

66. $a_n = \int_1^n x^{-2}\,dx$
67. $a_n = \frac{75^{n-1}}{99^n} + \frac{5^n \sin n}{8^n}$
68. $a_n = \tan^{-1}\left(\frac{10n}{10n + 4}\right)$
69. $a_n = \cos(0.99^n) + \frac{7^n + 9^n}{63^n}$

70–74. Sequences by recurrence relations *Consider the following sequences defined by a recurrence relation. Use a calculator, analytical methods, and/or graphing to make a conjecture about the value of the limit or determine that the limit does not exist.*

70. $a_{n+1} = \frac{1}{2} a_n + 2$; $a_0 = 5$, $n = 0, 1, 2, \dots$
71. $a_{n+1} = 2a_n(1 - a_n)$; $a_0 = 0.3$, $n = 0, 1, 2, \dots$
72. $a_{n+1} = \frac{1}{2}(a_n + 2/a_n)$; $a_0 = 2$, $n = 0, 1, 2, \dots$
73. $a_{n+1} = 4a_n(1 - a_n)$; $a_0 = 0.5$, $n = 0, 1, 2, \dots$
74. $a_{n+1} = \sqrt{2 + a_n}$; $a_0 = 1$, $n = 0, 1, 2, \dots$

75. Crossover point The sequence $\{n!\}$ ultimately grows faster than the sequence $\{b^n\}$ for any $b > 1$ as $n \to \infty$. However, b^n is generally greater than $n!$ for small values of n. Use a calculator to determine the smallest value of n such that $n! > b^n$ for each of the cases $b = 2$, $b = e$, and $b = 10$.

Applications

76. Fish harvesting A fishery manager knows that her fish population naturally increases at a rate of 1.5% per month while 80 fish are harvested each month. Let F_n be the fish population after the nth month, where $F_0 = 4000$ fish.

a. Write out the first five terms of the sequence $\{F_n\}$.
b. Find a recurrence relation that generates the sequence $\{F_n\}$.

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c. Does the fish population decrease or increase in the long run?
d. Determine whether the fish population decreases or increases in the long run if the initial population is 5500 fish.
e. Determine the initial fish population F_0 below which the population decreases.

T 77. The hungry hippo problem A pet hippopotamus weighing 200 lb today gains 5 lb per day with a food cost of 45¢/day. The price for hippos is 65¢/lb today but is falling 1¢/day.

a. Let h_n be the profit in selling the hippo on the nth day, where $h_0 = (200 \text{ lb}) \times (\$0.65) = \$130$. Write out the first ten terms of the sequence $\{h_n\}$.
b. How many days after today should the hippo be sold to maximize the profit?

T 78. Sleep model After many nights of observation, you notice that if you oversleep one night you tend to undersleep the following night and vice versa. This pattern of compensation is described by the relationship

$$x_{n+1} = \frac{1}{2}(x_n + x_{n-1}) \quad \text{for } n = 1, 2, 3, \ldots,$$

where x_n is the number of hours of sleep you get on the nth night and $x_0 = 7$ and $x_1 = 6$ are the number of hours of sleep on the first two nights, respectively.

a. Write out the first six terms of the sequence $\{x_n\}$ and confirm that the terms alternately increase and decrease.
b. Find an explicit expression for the nth term of the sequence.
c. What is the limit of the sequence?

T 79. Calculator algorithm The CORDIC (COordinate Rotation DIgital Calculation) algorithm is used by most calculators to evaluate trigonometric and logarithmic functions. An important number in the CORDIC algorithm, called the *aggregate constant*, is

$$\prod_{n=0}^{\infty} \frac{2^n}{\sqrt{1+2^{2n}}}, \text{ where } \prod_{n=0}^{k} a_n \text{ represents the product } a_0 \cdot a_1 \cdots a_k.$$

This infinite product is the limit of the sequence

$$\left\{ \prod_{n=0}^{0} \frac{2^n}{\sqrt{1+2^{2n}}}, \prod_{n=0}^{1} \frac{2^n}{\sqrt{1+2^{2n}}}, \prod_{n=0}^{2} \frac{2^n}{\sqrt{1+2^{2n}}}, \ldots \right\}.$$

(See the Guided Projects.) Estimate the value of the aggregate constant.

Additional Exercises

80. Bounded monotonic proof Prove that the drug dose sequence in Example 5,

$$d_{n+1} = 0.5 d_n + 100, \quad \text{for } n = 1, 2, 3, \ldots, d_1 = 100,$$

is bounded and monotonic.

T 81. Repeated square roots Consider the expression $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$, where the process continues indefinitely.

a. Show that this expression can be built in steps using the recurrence relation $a_0 = 1, a_{n+1} = \sqrt{1 + a_n}$ for $n = 0, 1, 2, 3, \ldots$. Explain why the value of the expression can be interpreted as $\lim_{n\to\infty} a_n$.
b. Evaluate the first five terms of the sequence $\{a_n\}$.
c. Estimate the limit of the sequence. Compare your estimate with $(1 + \sqrt{5})/2$, a number known as the *golden mean*.
d. Assuming the limit exists, use the method of Example 5 to determine the limit exactly.
e. Repeat the above analysis for the expression $\sqrt{p + \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}}$, where $p > 0$. Make a table showing the approximate value of this expression for various values of p. Does the expression seem to have a limit for all positive values of p?

T 82. A sequence of products Find the limit of the sequence

$$\{a_n\}_{n=2}^{\infty} = \left\{ \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{n}\right) \right\}.$$

T 83. Continued fractions The expression

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}$$

where the process continues indefinitely, is called a *continued fraction*.

a. Show that this expression can be built in steps using the recurrence relation $a_0 = 1, a_{n+1} = 1 + 1/a_n$ for $n = 0, 1, 2, 3, \ldots$. Explain why the value of the expression can be interpreted as $\lim_{n\to\infty} a_n$.
b. Evaluate the first five terms of the sequence $\{a_n\}$.
c. Using computation and/or graphing, estimate the limit of the sequence.
d. Assuming the limit exists, use the method of Example 5 to determine the limit exactly. Compare your estimate with $(1 + \sqrt{5})/2$, a number known as the *golden mean*.
e. Assuming the limit exists, use the same ideas to determine the value of

$$a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \ddots}}}}$$

where a and b are positive real numbers.

T 84. Towers of powers For a positive real number p, how do you interpret $p^{p^{p^{\cdot^{\cdot}}}}$, where the tower of exponents continues indefinitely? As it stands, the expression is ambiguous. The tower could be built from the top or from the bottom; that is, it could be evaluated by the recurrence relations

(1) $a_{n+1} = p^{a_n}$ (building from the bottom) or

(2) $a_{n+1} = a_n^p$ (building from the top),

where $a_0 = p$ in either case. The two recurrence relations have very different behaviors that depend on the value of p.

a. Use computations with various values of $p > 0$ to find the values of p such that the recurrence relation (2) has a limit. Find the maximum value of p for which the recurrence relation has a limit.
b. Show that recurrence relation (1) has a limit for certain values of p. Make a table showing the approximate value of the tower for various values of p. Estimate the maximum value of p for which the recurrence relation has a value.

85. Fibonacci sequence The famous Fibonacci sequence was proposed by Leonardo Pisano, also known as Fibonacci, in about A.D. 1200 as a model for the growth of rabbit populations. It is given by the recurrence relation $f_{n+1} = f_n + f_{n-1}$, for $n = 1, 2, 3, \ldots$, where $f_0 = f_1 = 1$. Each term of the sequence is the sum of its two predecessors.

a. Write out the first ten terms of the sequence.
b. Is the sequence bounded?
c. Estimate or determine $\varphi = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$, the ratio of successive terms of the sequence. Provide evidence that $\varphi = (1 + \sqrt{5})/2$, a number known as the *golden mean*.
d. Verify the remarkable result that

$$f_n = \frac{1}{\sqrt{5}}(\varphi^n - (-1)^n \varphi^{-n})$$

86. Arithmetic-geometric mean Pick two positive numbers a_0 and b_0 with $a_0 > b_0$ and write out the first few terms of the two sequences $\{a_n\}$ and $\{b_n\}$:

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad \text{for } n = 0, 1, 2 \ldots.$$

(Recall that the arithmetic mean $A = (p + q)/2$ and the geometric mean $G = \sqrt{pq}$ of two positive numbers p and q satisfy $A \geq G$.)

a. Show that $a_n > b_n$ for all n.
b. Show that $\{a_n\}$ is a decreasing sequence and $\{b_n\}$ is an increasing sequence.
c. Conclude that $\{a_n\}$ and $\{b_n\}$ converge.
d. Show that $a_{n+1} - b_{n+1} < (a_n - b_n)/2$ and conclude that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. The common value of these limits is called the arithmetic-geometric mean of a_0 and b_0, denoted $\text{AGM}(a_0, b_0)$.
e. Estimate AGM(12, 20). Estimate Gauss' constant $1/\text{AGM}(1, \sqrt{2})$.

87. The hailstone sequence Here is a fascinating (unsolved) problem known as the hailstone problem (or the Ulam Conjecture or the Collatz Conjecture). It involves sequences in two different ways. First, choose a positive integer N and call it a_0. This is the *seed* of a sequence. The rest of the sequence is generated as follows: For $n = 0, 1, 2, \ldots$

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even} \\ 3a_n + 1 & \text{if } a_n \text{ is odd} \end{cases}$$

However, if $a_n = 1$ for any n, then the sequence terminates.

a. Compute the sequence that results from the seeds $N = 2, 3, 4, \ldots, 10$. You should verify that in all these cases, the sequence eventually terminates. The hailstone conjecture (still unproved) states that for all positive integers N, the sequence terminates after a finite number of terms.
b. Now define the hailstone sequence $\{H_k\}$, which is the number of terms needed for the sequence $\{a_n\}$ to terminate starting with a seed of k. Verify that $H_2 = 1, H_3 = 7$, and $H_4 = 2$.
c. Plot as many terms of the hailstone sequence as is feasible. How did the sequence get its name? Does the conjecture appear to be true?

88. Prove that if $\{a_n\} \ll \{b_n\}$ (as used in Theorem 8.6), then $\{ca_n\} \ll \{db_n\}$, where c and d are positive real numbers.

QUICK CHECK ANSWERS

1. (a) bounded, monotonic; (b) bounded, not monotonic; (c) not bounded, not monotonic; (d) bounded, monotonic (both nonincreasing and nondecreasing). 2. If $r = -1$, the sequence is $\{-1, 1, -1, 1, \ldots\}$, the terms alternate in sign, and the sequence diverges. If $r = 1$, the sequence is $\{1, 1, 1, 1, \ldots\}$, the terms are constant, and the sequence converges. 3. Both changes would increase the steady-state level of drug. 4. $\{n^{1.1}\}$ grows faster; the limit is 0.

8.3 Infinite Series

We begin our discussion of infinite series with *geometric series*. These series arise more frequently than any other infinite series, they are used in many practical problems, and they illustrate all the essential features of infinite series in general. First let's summarize some important ideas from Section 8.1.

▸ The sequence of partial sums may be visualized nicely as follows:

$$\underbrace{\underbrace{\underbrace{a_1}_{S_1} + a_2}_{S_2} + a_3}_{S_3} + a_4 + \cdots$$

Recall that every infinite series $\sum_{k=1}^{\infty} a_k$ has a sequence of partial sums

$$S_1 = a_1, \qquad S_2 = a_1 + a_2, \qquad S_3 = a_1 + a_2 + a_3,$$

where in general $S_n = \sum_{k=1}^{n} a_k$, for $n = 1, 2, 3, \ldots$.